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A Piecewise Linear Approximation Method to Solve Fuzzy Separable Quadratic Programming Problem

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Abstract

This paper presents a piecewise linear approximation method for solving separable quadratic programming problems by using linear fuzzy goal programming (FGP) methodology. In the proposed approach, the objectives are first described fuzzily by introducing imprecise aspiration level to each of them. The fuzzy goals are then characterized by their associated membership functions for representation of goal achievement in terms of membership values of fuzzy goals.In the model formulation of the problem, the defined membership functions are first transformed into membership goals by assigning the highest membership value (unity) and introducing under- and over-deviational variables to each of them. Then, the membership goals in quadratic form are transformed into linear goals by using piecewise linear approximation method. In the solution process, minimization of under- deviational variables in the goal achievement function under the minsum FGP solution approach is considered. To illustrate the proposed approach a numerical example is solved. The model solution is also compared with the solution achieved by using Taylor series approximation method.

Keywords

Fuzzy Programming, Goal Programming, Fuzzy Goal Programming, Piecewise Linear Approximation.

1. Introduction

Most of the real-world decision problems are multiobjective in nature and they conflict to each other regarding optimization of objectives. To resolve the conflict, the goal programming (GP) approach has been introduced by Charnes and Cooper [1] in 1961. The main problem of using GP is that a precise aspiration level need be assigned for each of the objectives. But, in a real-life decision situation, it is difficult to set precise target values to objectives due to imprecise in nature of human judgments. To overcome such a situation, fuzzy programming (FP) approach has been introduced by Bellman and Zadeh [2] in 1970. Zimmermann first proposed the fuzzy linear programming [3] approach in 1978. In FP, membership functions are defined on the basis of assigned aspiration levels and tolerance ranges defined for the fuzzy goals. But, it is difficult to define tolerance ranges in a highly sensitive decision situation. To overcome such difficulties, goal programming approach in fuzzy environment has been first introduced by Narashimann [4] in 1980. Thereafter, FGP has been studied extensively [5] by the active researchers and has been applied to different real life problems [6, 7].

There are many real-world decision problems in different structural optimization areas. It is found that objectives of most of the industrial problems are nonlinear in nature. To solve such problems, different classical approaches have been developed and widely circulated in the literature. One of most widely used approaches is linear approximation of a nonlinear function. Two prominent methods of approximation of functions are Taylor Series approximation method and piecewise linear approximation method.

Taylor series approximation method is one of the most widely used for linearization of quadratic objectives. Taylor series approximation method has been used by Toksari [8], Pal and Moitra in [9] to the problems with fractional and quadratic objectives. But, they are rough approximations methods, and round-off errors may occur in practical decision situations.

Separable programming is a special branch of nonlinear programming where the objectives are expressed as the sum of separable functions of single variable. Separable programming was first introduced by Miller [10] in 1963. Thereafter, it has been developed by Cox [11], Keha et al. [12], Lin and Chen [13], Chang [14], Zhang and Wang [15], Croxton et al. [16], and other researchers. But, separable programming approach in the area of FGP is not widely circulated in literature. Generally, separable programming problems are transformed into linear form by using piecewise linear approximation method [17]. Since piecewise linear approximation approach approximates functions in a piecewise manner, the error estimation becomes less in comparison to the Taylor series approximation method.

In this paper, the piecewise linear approximation method are addressed to solve fuzzy multiobjective quadratic programming problem. In the model formulation of the problem, first the defined fuzzy goals are characterized algebraically by introducing associated tolerance limits. The membership functions are then converted into membership goals by taking maximum attainable membership value (unity) as the target level and introducing under-and over-deviational variables to each of them. In the solution process, the nonlinear membership goals in quadratic form are approximated to linear goals by using piecewise linear approximation method. The *minsum* GP methodology is employed to solve the problem.

The potential use of the approach is illustrated by a numerical example.

2. Problem formulation

The generic form multiobjective separable programming problem can be presented as:

$$\text{Max } Z_k(X) = \sum_{j=1}^{n} f_{kj}(x_j), \quad k = 1,2,\ldots,K_1$$

$$\text{Min } Z_k(X) = \sum_{j=1}^{n} f_{kj}(x_j), \quad k = (K_1+1),(K_1+2),\ldots,K$$

subject to

$$\sum_{j=1}^{n} c_{ij}x_j \le b_i, \; i = 1,2,\ldots,m$$

$$l_j \le x_j \le u_j, \quad j = 1,2,\ldots,n \qquad (1)$$

where $f_{kj}(x_j)$ represents univariate quadratic function of x_j and defined as

$$f_{kj}(x_j) = \alpha_{kj}x_j^2 + \beta_{kj}x_j + \gamma_{kj}$$

$k = 1,2,\ldots,K;\ j = 1,2,\ldots,n$ and where $\alpha_{kj}, \beta_{kj}, \gamma_{kj} \in R$

Let b_k be the imprecise aspiration level of the k-th objective $Z_k(X)$ (k=1, 2,….,K).

Then the fuzzy goals take the form as:

$Z_k(X) \gtrsim b_k$, k=1, 2,….,K_1

$Z_k(X)\ .\ b_k$ k= (K_1+1), (K_1+2),…., K

where X is the vector of decision variables, $\gtrsim$ and . represent the fuzziness of $\ge$ and $\le$ restrictions respectively, in the sense of Zimmermann [3].

The Fuzzy goals are characterized by their membership functions. The membership function for $\mu_k(X) \gtrsim b_k$ appear as [5]:

$$\mu_k(X) = \begin{cases} 1 & \text{if } Z_k(X) \ge b_k, \\ \dfrac{Z_k(X) - l_k}{b_k - l_k} & \text{if } l_k \le Z_k(X) < b_k, \\ 0 & \text{if } Z_k(X) < l_k, \end{cases}$$

$$k = 1, 2, \ldots, K_1 \qquad (2)$$

Again, for type of restriction, $\mu_k(X)$. b_k takes the form

$$\mu_k(X) = \begin{cases} 1 & \text{if } Z_k(X) \le b_k, \\ \dfrac{u_k - Z_k(X)}{u_k - b_k} & \text{if } b_k < Z_k(X) \le u_k, \\ 0 & \text{if } Z_k(X) > u_k, \end{cases}$$

$$k = (K_1+1), (K_1+2), \ldots, K. \qquad (3)$$

3. Fuzzy goal programming formulation

In the FGP model formulation of the problem, the defined membership functions are converted into the membership goals by introducing under- and over-deviational variables and assigning the highest membership value (unity) as the aspiration level to each of them. Then, the membership goals can be presented as:

$$\frac{Z_k(X) - l_k}{b_k - l_k} + d_k^- - d_k^+ = 1, \; k = 1, 2, \ldots, K_1 \qquad (4)$$

and

$$\frac{u_k - Z_k(X)}{u_k - b_k} + d_k^- - d_k^+ = 1 \qquad (5)$$

$$k = (K_1+1), (K_1+2), \ldots, K.$$

where $d_k^+, d_k^- \ge 0$ represent the under- and over-deviational variables concerned with achievement of the aspired level of the k-th membership goal.

In fuzzy goal programming formulation, since maximum value of the membership function has been set as aspiration level, only under-deviation associated with the respective goals are to be minimized to achieve the desired solution.

The weighted FGP model can be presented as:

Minimize $Z=\sum_{k=1}^{K} w_k d_k^-$,

and satisfy the goal expressions in (4), (5) and subject to the system constraints in (1), where $w_k (>0)$ represents the weight of importance of unwanted deviational variable associated with the kth goal and defined as [5]:

$$w_k = \begin{cases} \dfrac{1}{b_k - l_k}, \text{ if } \mu_k \text{ is defined as in (2)} \\ \dfrac{1}{u_k - b_k}, \text{ if } \mu_k \text{ is defined as in (3)} \end{cases}$$

4. Piecewise linearization of quadratic Goal

The goals in (4) and (5) can be explicitly expressed as:

$$\frac{1}{b_k - l_k}[\sum_{j=1}^{n} f_{kj}(x_j) - l_k] + d_k^- - d_k^+ = 1,\ k = 1,2,\ldots,K_1$$

and

$$\frac{1}{u_k - b_k}[u_k - \sum_{j=1}^{n} f_{kj}(x_j)] + d_k^- - d_k^+ = 1$$

$$k = (K_1+1), (K_1+2), \ldots, K$$

respectively.

To linearize the quadratic function $f_{kj}(x_j)$, the grid points (break points) for the variable x_j $(j=1,2,\ldots,n)$ are chosen as a_{jp} $(p = 0,1,\ldots,p_j)$. Introducing new variables y_{jp} $(p = 0,1,\ldots,p_j)$, x_j can be expressed as:

$$x_j = \sum_{p=1}^{p_j} a_{jp} y_{jp}$$

where $\sum_{p=0}^{p_j} y_{jp} = 1$ ($y_{jp} \geq 0$) with $a_{jo} = l_j$ and $a_{jp_j} = u_j$.

Then, the piecewise approximated linear form of the quadratic function $f_{kj}(x_j)$ (designated as F_{kj}) can be expressed as:

$$F_{kj} = \sum_{p=0}^{p_j} y_{jp} f_{kj}(a_{jp}) \qquad (6)$$

Using the relation in (6), the executable linear FGP model can be presented as:

Minimize $Z=\sum_{k=1}^{K} w_k d_k^-$

so as to satisfy

$$\frac{1}{b_k - l_k}[\sum_{j=1}^{n} F_{kj} - l_k] + d_k^- - d_k^+ = 1,$$

$$k = 1,2,\ldots,K_1$$

and

$$\frac{1}{u_k - b_k}[u_k - \sum_{j=1}^{n} F_{kj}] + d_k^- - d_k^+ = 1,$$

$$k = (K_1+1), (K_1+2), \ldots, K$$

where $F_{kj}(x_j) = \sum_{p=0}^{p_j} y_{jp} f_{kj}(a_{jp})$

subject to $\sum_{j=1}^{n} c_{ij} (\sum_{p=1}^{p_j} y_{jp} a_{jp}) \leq b_i$, $i = 1,2,\ldots,m$

$y_{jp} \geq 0$ $(j = 1,2,\ldots,n;\ p = 1,2,\ldots,p_j)$.

At most two y_{jp} may be positive and if two are positive, they must be consecutive.

To enforce the above condition binary variable z_{jp} $(j = 1,2,\ldots,n;\ p = 0,1,\ldots,p_j - 1)$ are to be introduced. The required restrictions appear as:

$$y_{j0} \leq z_{j0}$$

$$y_{jp} \leq z_{jp\text{-}1} + z_{jp},\ \forall p \in \{1,2,\ldots,p_j - 1\}$$

$$y_{jp_j} \leq z_{jp_j - 1}$$

$$\sum_{j=0}^{p_j - 1} z_{jp} = 1 \qquad (7)$$

To illustrate the proposed approach, a numerical example is solved in Section 5.

5. Numerical example

The maximization problem of two separable quadratic objectives is stated as:

Maximize $Z_1(X) = x_1^2 + x_2^2 + 2x_1 + 1$

Maximize $Z_2(X) = x_1^2 + x_2^2 - 1$

Subject to $x_1 + x_2 \le 5$

$x_1 \ge 1$

$x_2 \ge 2$ (8)

Using the constraints in (8), the bounds of the variables can be obtained as

$1 \le x_1 \le 3$

$2 \le x_2 \le 4$ (9)

Evaluating the best and worst individual solution of the objectives, the fuzzy goals can be presented as:

$Z_1(X) \gtrsim 20$,

$Z_2(X) \gtrsim 16$

Lower tolerances of the objectives are 15 and 14, respectively.

Now, membership functions are obtained as:

$\mu_1(X) = (1/5)[Z_1(x) - 15]$

$\mu_2(X) = (1/2)[Z_2(x) - 14]$

Each of the objective functions can be expressed as the sum of separable functions which are shown in the Table 1.

Table 1: Separable functions associated with the objectives

$f_{11}(x_1)$	$x_1^2 + 2x_1$
$f_{12}(x_2)$	$x_2^2 + 1$
$f_{21}(x_1)$	x_1^2
$f_{22}(x_2)$	$x_2^2 - 1$

Introducing under-and over-deviational variables, the membership goals can be expressed as:

$$\frac{1}{5}[f_{11}(x_1) + f_{12}(x_2) - 15] + d_1^- + d_1^+ = 1$$

$$\frac{1}{2}[f_{21}(x_1) + f_{22}(x_2) - 14] + d_2^- + d_2^+ = 1 \quad (10)$$

The membership goals are quadratic in nature. These functions are to be reduced in linear form.

To approximate the functions, the sets of grid points for the variables $x_1$and x_2 are {1, 1.5, 2, 2.5, 3} and {2, 2.5, 3, 3.5, 4}, respectively.

Using the proposed procedure the executable FGP model can presented as:

Minimize $Z = \frac{1}{5} d_1^- + \frac{1}{2} d_2^-$

so as to satisfy $\frac{1}{5}[F_{11} + F_{12} - 15] + d_1^- + d_1^+ = 1$

$\frac{1}{2}[F_{21} + F_{22} - 14] + d_2^- + d_2^+ = 1$

$F_{11} = 3y_{10} + 5.25y_{11} + 8y_{12} + 11.25y_{13} + 15y_{14}$

$F_{12} = 5y_{20} + 7.25y_{21} + 10y_{22} + 13.25y_{23} + 17y_{24}$

$F_{21} = y_{10} + 2.25y_{11} + 4y_{12} + 6.25y_{13} + 9y_{14}$

$F_{22} = 3y_{20} + 5.25y_{21} + 8y_{22} + 11.25y_{23} + 15y_{24}$

subject to

$y_{10} + 1.5y_{11} + 2y_{12} + 2.5y_{13} + 3y_{14} +$

$y_{20} + 2.5y_{21} + 3y_{22} + 3.5y_{23} + 4y_{24} \le 5$

$y_{10} + 1.5y_{11} + 2y_{12} + 2.5y_{13} + 3y_{14} \ge 1$

$y_{20} + 2.5y_{21} + 3y_{22} + 3.5y_{23} + 4y_{24} \ge 2$

$y_{10} + y_{11} + y_{12} + y_{13} + y_{14} = 1$

$y_{20} + y_{21} + y_{22} + y_{23} + y_{24} = 1$

$y_{10} \le z_{10}$

$y_{11} \le z_{10} + z_{11}$

$y_{12} \le z_{11} + z_{12}$

$y_{13} \le z_{12} + z_{13}$

$y_{14} \le z_{13}$

$y_{20} \le z_{20}$

$y_{21} \le z_{20} + z_{21}$

$y_{22} \le z_{21} + z_{22}$

$y_{23} \le z_{22} + z_{23}$

$y_{24} \le z_{23}$

where $y_{jp} \ge 0$ $(j = 1, 2; p = 0, 1, 2, 3, 4)$ and z_{jp} $(j = 1, 2; p = 0, 1, 2, 3)$ are binary variables and satisfy the conditions presented as:

$z_{10} + z_{11} + z_{12} + z_{13} = 1$

$z_{20} + z_{21} + z_{22} + z_{23} = 1$

Using the software LINGO (Version 6.0) the problem is solved and the obtained solution is

$(x_1, x_2) = (1, 4)$ with $(Z_1, Z_2) = (20, 16)$

Then, the resulting membership values are achieved as:

$\mu_1 = 1, \ \mu_2 = 1$

The results show that the solution is achieved according to the needs and desires of the decision maker in the imprecise decision environment.

Note: The function $Z_k(X)$ can be approximated by using first order Taylor series approximation method about the point $X^{\bullet}(x_1^*, x_2^*, \ldots, x_n^*)$ as:

$$Z_k(x_1, x_2, \ldots, x_n) \approx Z_k(x_1^*, x_2^*, \ldots, x_n^*) + \sum_{j=1}^{n} (x_j - x_j^*) \frac{\partial Z_k(x_1^*, x_2^*, \ldots, x_n^*)}{\partial x_j}$$

Now using first order Taylor series approximation about the point (3, 2), the linear equivalent goals of the quadratic goals in (10) can be presented as:

$$(1/5)[8x_1 + 4x_2 - 27] + d_1^- + d_1^+ = 1$$

$$(1/2)[6x_1 + 4x_2 - 28] + d_2^- + d_2^+ = 1 \qquad (11)$$

Then the executable FGP model can be expressed as:

Minimize $Z = \frac{1}{5} d_1^- + \frac{1}{2} d_2^-$

so as to satisfy the goal equations in (11) and subject to the set of constraints in (8).

Using the software LINGO (Version 6.0) the problem is solved and the solution is obtained as:

$(x_1, x_2) = (3, 2)$ with $(Z_1, Z_2) = (20, 12)$

The comparison of the objective values, obtained by using the proposed approach and first order Taylor series approximation, is shown in the Figure 1.

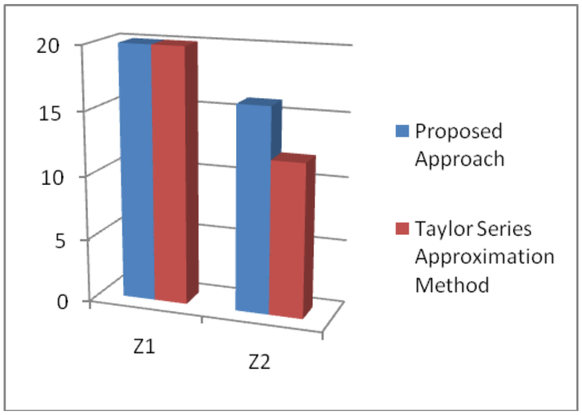

Figure 1: Comparison of the objective values obtained by two different approaches.

The Comparisons show that the solution achieved under the proposed approach is superior over the Taylor series approximation approach.

6. Conclusion

In this paper, piecewise linear approximation method has been employed for solving fuzzy separable quadratic programming problem. Under the proposed approach, the accuracy of solution can easily be made where needed by increasing the number of grid points without involving any computational complexity.

The proposed method can be extended to solve linear fractional programming problems, quadratic fractional programming problems and bilevel programming problems with separable objective functions as well as constraints, which may be the problems in future studies.

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