International Journal of Advanced Computer Research (ISSN (print): 2249-7277 ISSN (online): 2277-7970)
Volume-3 Number-3 Issue-11 September-2013

Perturbation of n-Dimensional Quadratic Functional Equation: A Fixed Point Approach

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Abstract

In this paper, the authors investigate the generalized Ulam-Hyers stability of $n-$ dimensional quadratic functional equation

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i) = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left(g(x_j+x_{i+1}) - g(x_j - x_{i+1})\right)$$

with $n \geq 2$ with the help of fixed point method. An application of this functional equation is also discussed.

Keywords

Quadratic functional equation, Generalized Ulam-Hyers stability, JM Rassias stability.

1. Introduction

During the last seven decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 2, 3, 4, 5, 6, 7]. The terminology generalized Ulam - Hyers stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. For more detailed definitions of such terminologies, one can refer to [8, 9, 10, 11, 12, 13, 14].

The solution and stability of following quadratic functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \qquad \text{(I.1)}$$

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(x+z), \qquad \text{(I.2)}$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x), \qquad \text{(I.3)}$$

$$f(x+y+z) + f(x-y) + f(y-z) + f(z-x) = 3f(x) + 3f(y) + 3f(z), \qquad \text{(I.4)}$$

$$f(2x \pm y \pm z) + 2f(y) + 2f(z) = 2f(x \pm y) + 2f(x \pm z) + f(y+z) \qquad \text{(I.5)}$$

were investigated by S.Czerwik [9], S.M. Jung [15], PL.Kannappan [16], Y.H. Bae, K.W. Jun [17], M.Arunkumar et al., [18] and I.S. Chang, H.M. Kim [20].

Recently, M.Arunkumar et. al., [19] introduced and investigate the general solution and generalized Ulam-Hyers stability of a $n-$ dimensional quadratic functional equation

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i) = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left(g(x_j+x_{i+1}) - g(x_j - x_{i+1})\right) \qquad \text{(I.6)}$$

with $n \geq 2$.

In this paper, the authors studied the stability of the above functional equation (I.6) using fixed point approach.

2. Fixed Point Stability Results

In this section, the authors proved the generalized Ulam - Hyers stability of the n-dimensional quadratic functional equation (I.6) in Banach spaces with the help of fixed point method.

Now we will recall the fundamental results in fixed point theory (see [22, 21]).

Theorem 2.1: ***(Banach's contraction principle)*** *Let* (X, d) *be a complete metric space and consider a mapping* $T: X \to X$ *which is strictly contractive mapping, that is*

(A_1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*

(i) The mapping T *has one and only fixed point* $x^* = T(x^*)$;

(ii) The fixed point for each given element x^* *is globally attractive, that is*

(A_2) $\lim_{n\to\infty} T^n x = x^*$, *for any starting poin tx in X;*

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(iii) One has the following estimation inequalities:

(A_3) $d(T^n x, x^*) \le \frac{1}{1-L}\ d(T^n x, T^{n+1}x)$, *for all* $n \ge 0, x \in X$.

(A_4) $d(x, x^*) \le \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X.$

Theorem 2.2: [21] ***(The alternative of fixed point)*** *Suppose that for a complete generalized metric space* (X, d) *and a strictly contractive mapping* $T : X \to X$ *with Lipschitz constant* L. *Then, for each given element* $x \in X$, *either*

(B_1) $d(T^n x, T^{n+1}x) = \infty\ \ \forall\ \ n \ge 0,$ *or*

(B_2) *there exists a natural number* n_0 *such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \ge n_0$;

(ii) *The sequence* $(T^n x)$ *is convergent to a fixed point* y^* *of* T

(iii) y^* *is the unique fixed point of* T *in the set* $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;

(iv) $d(y^*, y) \le \frac{1}{1-L}\ d(y, Ty)$ *for all* $y \in Y$.

In this section, assume X be a normed space and Y be a Banach space. For proving the stability result we define the following:

Let Ψ be a mapping from X^n to Y defined by

$$\Psi(x) = \Psi(x_1, x_2, x_3, \ldots, x_n) = \sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i) - \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\big(g(x_j + x_{i+1}) - g(x_j - x_{i+1})\big)$$

for all $x \in X^n$ and $n \ge 2$, ς_i is a constant such that

$$\varsigma_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases},$$

and Ω is the set such that

$$\Omega = \{g \mid g : X \to Y, g(0) = 0\}.$$

The following theorem provide the stability result of (I.6) using fixed point method.

Theorem 2.3: *Let* $g : X \to Y$ *be a mapping for which there exists a function* $\phi : X^n \to [0, \infty)$ *with the condition*

$$\lim_{k\to\infty} \frac{\phi(\varsigma_i^k x)}{\varsigma_i^{2k}} = 0, \qquad (II.1)$$

satisfying the functional inequality

$$\|\Psi(x)\| \le \phi(x) \qquad (II.2)$$

for all $x = (x_1, x_2, x_3, \ldots, x_n) \in X^n$ *and* $n \ge 2$. *If there exists* $L = L(i) < 1$ *such that the function*

$$x \to \beta(x) = \frac{1}{2(n-1)}\ \phi\left(\frac{x}{2}, \frac{x}{2}, \underbrace{0, \ldots, 0}_{n-2\ times}\right),$$

has the property

$$\frac{\beta(\varsigma_i x)}{\varsigma_i^2} \le L\beta(x) \quad \forall\ x \in X. \qquad (II.3)$$

Then there exists a unique quadratic function $Q : X \to Y$ *satisfying the functional equation (I.6) and*

$$\|g(x) - Q(x)\| \le \frac{L^{1-i}}{1-L}\beta(x) \quad \forall\ x \in X. \qquad (II.4)$$

Proof. Let d be a general metric on Ω, such that

$$d(g, h) = inf\left\{K \in (0, \infty) : \|g(x) - h(x)\| \le K\beta(x), x \in X\right\}.$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \to \Omega$ by $Tg(x) = \frac{1}{\varsigma_i^2} g(\varsigma_i x)$, for all $x \in X$. For $g, h \in \Omega$ and $x \in X$, we have

$$\begin{aligned}
d(g,h) = K &\Rightarrow \|g(x) - h(x)\| \le K\beta(x) \\
&\Rightarrow \left\|\frac{g(\varsigma_i x)}{\varsigma_i^2} - \frac{h(\varsigma_i x)}{\varsigma_i^2}\right\| \le \frac{1}{\varsigma_i^2} K\beta(\varsigma_i x) \\
&\Rightarrow \|Tg(x) - Th(x)\| \le \frac{1}{\varsigma_i^2} K\beta(\varsigma_i x) \\
&\Rightarrow \|Tg(x) - Th(x)\| \le LK\beta(x) \\
&\Rightarrow d(Tg(x), Th(x)) \le KL \\
&\Rightarrow d(Tg, Th) \le Ld(g, h).
\end{aligned}$$

Therefore T is strictly contractive mapping on Ω with Lipschitz constant L. Replacing

$x=(x_1,x_2,x_3,\dots,x_n)$ by $(x,x,\underbrace{0,\dots,0}_{n-2\ times})$ in (II.2), we get

$$\|g(2x)-4g(x)\|\le \frac{1}{2(n-1)}\phi(x,x,\underbrace{0,\dots,0}_{n-2\ times}) \tag{II.5}$$

for all $x\in X$. Using the definition of $\beta(x)$ in the above equation and for $i=0$, we have

$$\left\|\frac{g(2x)}{4}-g(x)\right\|\le\frac{1}{4}\beta(2x)$$

i.e., $\|Tg(x)-g(x)\|\le L\beta(x)$

for all $x\in X$. Hence, we arrive

$$d(Tg(x)-g(x))\le L=L^{1-i}. \tag{II.6}$$

for all $x\in X$. Replacing x by $\frac{x}{2}$ in (II.5), we get

$$\left\|(g(x)-4g\left(\frac{x}{2}\right))\right\|\le\frac{1}{2(n-1)}\phi\left(\frac{x}{2},\frac{x}{2},\underbrace{0,\dots,0}_{n-2\ times}\right) \tag{II.7}$$

for all $x\in X$. Using the definition of $\beta(x)$ in the above equation and for $i=0$, we have

$$\left\|g(x)-4g\left(\frac{x}{2}\right)\right\|\le\beta(x)$$

i.e., $\|g(x)-Tg(x)\|\le\beta(x)$ for all $x\in X$.

Hence, we arrive

$$d(g(x)-Tg(x))\le 1=L^{1-i} \tag{II.8}$$

for all $x\in X$. From (II.6) and (II.8), we can conclude

$$d(g(x)-Tg(x))\le L^{1-i}<\infty. \tag{II.9}$$

for all $x\in X$. Now from the fixed point alternative in both cases, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x)=\lim_{k\to\infty}\frac{g\left(\varsigma_i^k x\right)}{\varsigma_i^{2k}},\forall x\in X. \tag{II.10}$$

In order to prove $Q:X\to Y$ satisfies the functional equation (I.6), Replace x by $\varsigma_i^k x$ and divide by ς_i^{2k} in (II.2), we arrive

$$\left\|\frac{1}{\varsigma_i^{2k}}\Psi(\varsigma_i^k x)\right\|\le\frac{1}{\varsigma_i^{2k}}\phi\left(\varsigma_i^k x\right) \tag{II.11}$$

for all $x\in X^n$. This implies that,

$$\Bigg\|\sum_{\ell=1}^{n}\frac{1}{\varsigma_i^{2k}}g\left(\sum_{j=1}^{\ell}\varsigma_i^k x_j\right)-\sum_{\ell=1}^{n}\frac{(n-\ell+1)}{\varsigma_i^{2k}}g\left(\varsigma_i^k x_\ell\right)$$
$$-\frac{1}{2\varsigma_i^{2k}}\sum_{\ell=1}^{n-1}(n-\ell)\sum_{j=1}^{\ell}[g(\varsigma_i^k x_j+\varsigma_i^k x_{\ell+1})$$
$$-g(\varsigma_i^k x_j-\varsigma_i^k x_{\ell+1})]\Bigg\|\le\frac{1}{\varsigma_i^{2k}}\phi\left(\varsigma_i^k x\right) \tag{II.12}$$

$x\in X^n$. Letting $k\to\infty$ in the above inequality and using the choice of Q and ϕ, we arrive

$$\sum_{\ell=1}^{n}Q\left(\sum_{j=1}^{\ell}x_j\right)-\sum_{\ell=1}^{n}(n-\ell+1)Q(x_\ell)$$
$$=\frac{1}{2}\sum_{\ell=1}^{n-1}(n-\ell)\sum_{j=1}^{\ell}\left[Q(x_j+x_{\ell+1})-Q(x_j-x_{\ell+1})\right] \tag{II.13}$$

$x\in X^n$. Hence Q satisfies the functional equation (I.6). Since Q is unique fixed point of T in the set

$$\Delta=\{g\in\Omega\,|\,d(g,Q)<\infty\},$$

therefore Q is a unique function such that

$$\|g(x)-Q(x)\|\le K\beta(x)\,\forall\,x\in X. \tag{II.14}$$

Again using the fixed point alternative, we obtain

$$d(g,Q)\le\frac{1}{1-L}d(g,Tg)$$

i.e., $d(g,Q)\le\dfrac{L^{1-i}}{1-L}$

i.e., $\|g(x)-Q(x)\|\le\dfrac{L^{1-i}}{1-L}\beta(x).$ (II.15)

for all $x\in X$. This completes the proof of the theorem.

The following corollaries are immediate consequence of Theorems 2.3 concerning the stability of (I.6).

Corollary 2.4 *Suppose that a function* $g:X\to Y$ *satisfies the inequality*

$$\|\Psi(x)\|\le\varepsilon\sum_{\ell=1}^{n}\|x_\ell\|^p \tag{II.16}$$

for all $x=(x_1,x_2,x_3,\dots,x_n)\in X^n$, *where* $\varepsilon>0, p\ne 2$ *are constants. Then there exists a unique quadratic mapping* $Q:X\to Y$ *such that*

$$\|g(x)-Q(x)\| \le \frac{\varepsilon}{(n-1)|4-2^p|}\|x\|^p \ \forall x \in X \qquad (II.17)$$

Proof. Let $\phi(x) = \varepsilon \sum_{\ell=1}^{n} \| x_\ell \|^p$

for all $x=(x_1,x_2,x_3,\ldots,x_n) \in X^n$. Then for $p<2, i=0$ and for $p>2, i=1$, we arrive

$$\frac{\phi(\varsigma_i^k x)}{\varsigma_i^{2k}} = \frac{\phi(\varsigma_i^k x_1, \varsigma_i^k x_2, \varsigma_i^k x_3, \ldots, \varsigma_i^k x_n)}{\varsigma_i^{2k}}$$

$$= \frac{\varepsilon}{\varsigma_i^{2k}} \sum_{\ell=1}^{n} \| \varsigma_i^k x_\ell \|^p = \varepsilon\, \varsigma_i^{(p-2)k} \sum_{\ell=1}^{n} \| x_\ell \|^p$$

$\to 0$ as $k \to \infty$.

Thus, (II.1) is holds. But we have

$$\beta(x) = \frac{1}{2(n-1)}\ \phi\left(\frac{x}{2}, \frac{x}{2}, \underbrace{0,\ldots,0}_{n-2\, times}\right)$$

has the property $\frac{1}{\varsigma_i^2}\beta(\varsigma_i x) = L\beta(x) \, \forall x \in X$.

Hence

$$\beta(x) = \frac{\varepsilon}{2(n-1)}\left(\left\|\frac{x}{2}\right\|^p + \left\|\frac{x}{2}\right\|^p\right) = \frac{2^{-p}\varepsilon}{(n-1)}\|x\|^p.$$

for all $x \in X$. Replace x by $\varsigma_i x$ and divide by ς_i^2 in above equality, we get

$$\frac{1}{\varsigma_i^2}\beta(\varsigma_i x) = \frac{2^{-p}\varepsilon}{(n-1)\varsigma_i^2}\|\varsigma_i x\|^p$$

$$= \varsigma_i^{p-2}\frac{2^{-p}\varepsilon}{(n-1)}\|x\|^p = \varsigma_i^{p-2}\beta(x)$$

for all $x \in X$. Hence the inequality (II.3) holds when, $L = \varsigma_i^{p-2}$, that is

$$L = \begin{cases} 2^{p-2} & for \ \ i=0, p<2, \\ 2^{2-p} & for \ \ i=1, p>2. \end{cases}$$

Now from (II.4), we prove the following cases:

Case:1 $L = 2^{p-2}$ for $p<2$ if $i=0$

$$\|g(x)-Q(x)\| \le \frac{L^{1-0}}{1-L}\beta(x)$$

$$= \frac{2^{p-2}}{(1-2^{p-2})}\frac{2^{-p}\varepsilon}{(n-1)}\|x\|^p$$

$$= \frac{\varepsilon}{(n-1)(4-2^p)}\|x\|^p$$

Case:2 $L = 2^{2-p}$ for $p>2$ if $i=1$

$$\|g(x)-Q(x)\| \le \frac{L^{1-1}}{1-L}\beta(x)$$

$$= \frac{1}{(1-2^{2-p})}\frac{2^{-p}\varepsilon}{(n-1)}\|x\|^p$$

$$= \frac{\varepsilon}{(n-1)(2^p-4)}\|x\|^p$$

From the above two cases we arrive (II.17). Hence the proof is complete.

Corollary 2.5 *Suppose that a function* $g: X \to Y$ *satisfies the inequality*

$$\|\Psi(x)\| \le \varepsilon\left\{\sum_{\ell=1}^{n} \| x_\ell \|^{np} + \prod_{\ell=1}^{n} \| x_\ell \|^p\right\} \qquad (II.18)$$

for all $x=(x_1,x_2,x_3,\ldots,x_n) \in X^n$, *where* ε, p *are constants with* $\varepsilon > 0, p \ne \frac{2}{n}$. *Then there exists a unique quadratic mapping* $Q: X \to Y$ *such that*

$$\|g(x)-Q(x)\| \le \frac{\varepsilon}{(n-1)|4-2^{np}|}\|x\|^{np} \qquad (II.19)$$

for all $x \in X$.

Proof. Let $\phi(x) = \varepsilon\left\{\sum_{\ell=1}^{n} \| x_\ell \|^{np} + \prod_{\ell=1}^{n} \| x_\ell \|^p\right\}$

for all $x=(x_1,x_2,x_3,\ldots,x_n) \in X^n$. Then for $p<\frac{1}{2}, i=0$ and for $p>\frac{1}{2}, i=1$, we arrive

$$\frac{\phi(\varsigma_i^k x)}{\varsigma_i^{2k}} = \frac{\phi(\varsigma_i^k x_1, \varsigma_i^k x_2, \varsigma_i^k x_3, \ldots, \varsigma_i^k x_n)}{\varsigma_i^{2k}}$$

$$= \frac{\varepsilon}{\varsigma_i^{2k}}\left\{\sum_{\ell=1}^{n} \| \varsigma_i^k x_\ell \|^{np} + \prod_{\ell=1}^{n} \| \varsigma_i^k x_\ell \|^p\right\}$$

$$= \varsigma_i^{(np-2)k}\varepsilon\left\{\sum_{\ell=1}^{n} \| x_\ell \|^{np} + \prod_{\ell=1}^{n} \| x_\ell \|^p\right\}$$

$\to 0$ as $k \to \infty$.

Thus, (1) is holds. But we have

$$\beta(x) = \frac{1}{2(n-1)}\ \phi\left(\frac{x}{2}, \frac{x}{2}, \underbrace{0,\ldots,0}_{n-2\, times}\right)$$

has the property $\frac{1}{\varsigma_i^2}\beta(\varsigma_i x) = L\beta(x) \forall x \in X$.

Hence

$$\beta(x) = \frac{\varepsilon}{2(n-1)}\left(\left\|\frac{x}{2}\right\|^{np} + \left\|\frac{x}{2}\right\|^{np}\right) = \frac{2^{2-np}\varepsilon}{(n-1)}\|x\|^{np}.$$

for all $x \in X$. Replace x by $\varsigma_i x$ and divide by ς_i^2 in the above equality, we get

$$\frac{1}{\varsigma_i^2}\beta(\varsigma_i x) = \frac{2^{-np}\varepsilon}{(n-1)\varsigma_i^2}\|\varsigma_i x\|^p = \varsigma_i^{np-2}\frac{2^{-np}\varepsilon}{(n-1)}\|x\|^p$$
$$= \varsigma_i^{np-2}\beta(x)$$

for all $x \in X$. Hence the inequality (II.3) holds when, $L = \varsigma_i^{np-2}$, that is

$$L = \begin{cases} 2^{np-2} \; for\, i = 0, p < \frac{1}{2}, \\ 2^{2-np} \; for\, i = 1, p > \frac{1}{2}. \end{cases}$$

Now from (II.4), we prove the following cases:

Case:1 $L = 2^{np-2}$ for $p < \frac{1}{2}$ if $i = 0$

$$\|g(x) - Q(x)\| \le \frac{L^{1-0}}{1-L}\beta(x)$$
$$= \frac{2^{np-2}}{(1-2^{np-2})}\frac{2^{-np}\varepsilon}{(n-1)}\|x\|^{np}$$
$$= \frac{\varepsilon}{(n-1)(4-2^{np})}\|x\|^{np}$$

Case:2 $L = 2^{2-np}$ for $p > \frac{1}{2}$ if $i = 1$

$$\|g(x) - Q(x)\| \le \frac{L^{1-1}}{1-L}\beta(x)$$
$$= \frac{1}{(1-2^{2-np})}\frac{2^{-np}\varepsilon}{(n-1)}\|x\|^{np}$$
$$= \frac{\varepsilon}{(n-1)(2^{np}-4)}\|x\|^{np}$$

From the above two cases we arrive (II.19). Hence the proof is complete.

Corollary 2.6 *Suppose that a function* $g : X \to Y$ *satisfies the inequality*

$$\|\Psi(x)\| \le \varepsilon \qquad \text{(II.20)}$$

for all $x = (x_1, x_2, x_3, \ldots, x_n) \in X^n$, where $\varepsilon > 0$ is a constant. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|g(x) - Q(x)\| \le \frac{\varepsilon}{3(n-1)} \qquad \text{(II.21)}$$

for all $x \in X$.

Proof. The proof of the corollary is similar tracing to that of above corollary, by taking

$$L = \begin{cases} 2^{-2} \quad for \;\; i = 0, p = 0, \\ 2^{2} \quad for \;\; i = 1, p = 0. \end{cases}$$

3. Application

Consider the quadratic functional equation (I.6), that is

$$\sum_{i=1}^{n} g\left(\sum_{j=1}^{i} x_j\right) - \sum_{i=1}^{n}(n-i+1)g(x_i)$$
$$= \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left(g(x_j + x_{i+1}) - g(x_j - x_{i+1})\right)$$

Since $g(x) = x^2$ is the solution of the functional equation, the above equation can be rewritten as follows

$$\sum_{i=1}^{n}\left(\sum_{j=1}^{i} x_j\right)^2 = \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left((x_j + x_{i+1})^2\right.$$
$$\left.-(x_j - x_{i+1})^2\right) + \sum_{i=1}^{n}(n-i+1)(x_i)^2$$

Now, let us take the variables as consecutive terms, we arrive that the partial sums of the consecutive terms is equal to the right hand side terms. Mathematically

$$[x_1]^2 + [x_1 + x_2]^2 + \cdots + [x_1 + x_2 + x_3 + \cdots + x_n]^2$$
$$= \frac{1}{2}\sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{i}\left([x_j + x_{i+1}]^2 - [x_j - x_{i+1}]^2\right)$$
$$+\left(n[x_1]^2 + (n-1)[x_2]^2 + \ldots + [x_n]^2\right).$$

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