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Robustness of triple I algorithms based on Schweizer-Sklar operators in fuzzy reasoning

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Abstract
In this paper, the perturbation of fuzzy connectives and the robustness of fuzzy reasoning are investigated. This perturbation of Schweizer-Sklar parameterized t-norms and its residuated implication operators are given. We show that full implication triple I algorithms based on Schweizer-sklar operators are robust for normalized Minkowski distance.

Keywords
Schweizer-Sklar operators, Triple I algorithms, Fuzzy reasoning, Robustness.

1.Introduction
Since Zadeh [18] proposed the concept of fuzzy sets in 1965, the fuzzy theory is widely used in mathematics and many other application fields. As for fuzzy reasoning, the most basic models are given as follows (see [16, 22]):

Given the input "x is A^*" and fuzzy rule "if x is A then y is B", try to deduce a reasonable output "y is B^*", fuzzy modus ponens (FMP);

Given the input "y is B^*" and fuzzy rule "if x is A then y is B", try to deduce a reasonable output "x is A^*", fuzzy modus tollens (FMT).

Zadeh [19, 20, 21] introduce an influential approach called compositional rule of inference (CRI method) to deal with FMP and FMT. But CRI method lacks solid logical basis and has some arbitrariness.

In 1999, Wang [12, 13] propose a triple I method for fuzzy reasoning based on R_0-implication operator by combining fuzzy logic and fuzzy reasoning, establishing the triple I principles for the models FMP and FMT.

Wang and Fu [15] established full implication triple I inference algorithms based on regular implications and normal implications. Pei [10] discuss the full implication inference for all residuated implication introduced by left continuous t-norms to solve FMP and FMT problem. In addition, Pei [11] conducted a detailed research into the triple I algorithms based on the monoidal t-norm basic logical system MTL setting a sound logic foundation. Furthermore, Luo and Yao [8] studied the full implication triple I algorithms based on Schweizer-Sklar parameterized family of t-norms and the continuity of this algorithm.

In fuzzy control, practical fuzzy reason schemes are likely to be perturbed by various types of noise. Therefore, the robustness of fuzzy inference algorithms is important. Cai [1] analyzed the robustness based on measuring the errors of consequents produced by the errors of premises in fuzzy reasoning based on equalities. Dai et al. [2] discuss the perturbation of some t-norms and corresponding residuated implication operators, and the robustness of the CRI solution for the FMP and FMT model. Subsequently [3] investigate the degree of equality of fuzzy sets and some important fuzzy implications and the robustness of triple I methods for fuzzy reasoning. Li et al. [9] studied the robustness of fuzzy reasoning by a concept similar to the modulus of continuity.

In this paper, the perturbation of Schweizer-Sklar operators and the robustness of triple I algorithms

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based on Schweizer-Sklar operators are investigated based on approximately equal under Minkowski distance. First, we review the necessary definitions and some lemmas relate to this paper. In section 3, the perturbation of Schweizer-Sklar operators is studied. In section 4, the robustness of triple I algorithms based on Schweizer-Sklar operators is discussed.

2.Preliminaries

In this section, the definitions of Schweizer-Sklar t-norms, their residuated implication operators and the solution of parameterized triple I algorithms are reviewed; in addition, some necessary lemmas are listed.

Definition 1 ([5, 7, 14]) The Schweizer-Sklar parameterized family of t-norms is a function $T_m:[0,1]^2\to[0,1]$, for all $x,y\in[0,1]$ and $m\in R$,

$$T_m(x,y)=\begin{cases}(\max(0,x^m+y^m-1))^{\frac{1}{m}},m\in(-\infty,0)\cup(0,\infty);\\ xy,\quad m=0\ (the\ product\ T_P);\\ \min\{x,y\},\quad m=-\infty\ (the\ \min imum\ T_G);\\ \begin{cases}\min\{x,y\},x\wedge y=1\\ 0,\quad otherwise\end{cases},m=\infty\\ (drastic\ product\ T_D).\end{cases}$$

Definition 2 ([5, 14]) The Schweizer-Sklar family of residuated implication operators induced by parameterized t-norms is also a function $I_m:[0,1]^2\to[0,1]$, for all $x,y\in[0,1]$ and $m\in R$,

$$I_m(x,y)=\begin{cases}\min\left\{1,(1-x^m+y^m)^{\frac{1}{m}}\right\},m\in(-\infty,0)\cup(0,\infty);\\ \min\left\{1,\frac{y}{x}\right\},m=0;\\ \begin{cases}1,x\le y\\ y,x>y\end{cases},m=-\infty;\\ \begin{cases}y,x=1\\ 1,x\ne 1\end{cases},m=\infty.\end{cases}$$

Definition 3 ([6]) The normalized Minkowski distance of fuzzy sets A and B is

$$d_p(A,B)=\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}|A(x_i)-B(y_i)|^p},$$

where p is a parameter satisfying.

Definition 4 ([2]) Let $U=\{x_1,x_2,\cdots\cdots x_n\}$ be a universe, A and B be two fuzzy sets defined on U, and $\varepsilon\in[0,1]$. If

$$d_p(A,B)=\sqrt[p]{\frac{1}{n}\sum_{i=1}^{n}|A(x_i)-B(x_i)|^p}\le\varepsilon,$$

then A and B are said to be $\varepsilon(p)$-approximately equal, denoted $A\approx_p(\varepsilon)B$. Also, B is called an $\varepsilon(p)$-perturbation of A.

We denote the residuated implication operators based on the Schweizer-Sklar t-norm by $\to_m$.

Lemma 1([8]) The $\to_m$ type triple I solution B^* for FMP is given by the following formula: $\forall\, y\in Y$, when $m<\infty$,

$$B^*(y)=\begin{cases}\sup\limits_{x\in E_y}\left\{((A(x)\to_m B(y))^m+(A^*(x))^m-1)^{\frac{1}{m}}\right\},\\ m\in(-\infty,0)\cup(0,\infty);\\ \sup\limits_{x\in E_y}\left\{((A(x)\to_p B(y))\cdot A^*(x)\right\},m=0;\\ \sup\limits_{x\in E_y}\left\{A^*(x)\wedge(A(x)\to_G B(y))\right\},m=-\infty;\end{cases}$$

Where

$$E_y=\begin{cases}\left\{x\in X\,\middle|\,(A(x)\to_m B(y))^m+(A^*(x))^m-1)>0\right\},\\ m\in(-\infty,0)\cup(0,\infty);\\ \left\{x\in X\,\middle|\,A^*(x)>0\ and\ A(x)\to_m B(y)>0\right\},\\ m=0,-\infty.\end{cases}$$

when $m=\infty$

$$B^*(y)=\begin{cases}\sup\limits_{x\in E_y}\left\{A^*(x)\wedge(A(x)\to_D B(y))\right\},\\ E_y=\left\{x\in X\,\middle|\,A^*(x)\vee(A(x)\to_D B(y))=1\right\};\\ 0,\\ E_y=\left\{x\in X\,\middle|\,A^*(x)\vee(A(x)\to_D B(y))\ne 1\right\}.\end{cases}$$

Lemma 2([8]) The $\to_m$ type triple I solution A^* B^* for FMT is given by the following formula: $\forall\, x\in X$, when $m<\infty$,

$$A^*(x)=\begin{cases}\inf_{y\in E_x}\{(1-(A(x)\to_m B(y))^m+(B^*(y))^m)^{\frac{1}{m}}\},\\ m\in(-\infty,0)\cup(0,\infty);\\ \inf_{y\in E_x}\{\frac{B^*(y)}{A(x)\to_p B(y)}\}, m=0;\\ \inf_{y\in E_x}\{B^*(y)\},\quad m=-\infty;\end{cases}$$

where $E_x=\{y\in Y\,|\,B^*(y)<A(x)\to_m B(y), m<\infty\}$;

when $m=\infty$,

$$A^*(x)=\begin{cases}\inf_{y\in E_x}\{B^*(y)\}, E_x=\{y\in Y\,|\,A(x)\to_m B(y)=1\};\\ 1, E_x=\{y\in Y\,|\,A(x)\to_m B(y)\neq 1\}.\end{cases}$$

Lemma 3 ([4]) Let $x,y>0$ and $x\neq y$, then

$rx^{r-1}(x-y)>x^r-y^r>ry^{r-1}(x-y)(r<0\ or\ r>1)$

$rx^{r-1}(x-y)<x^r-y^r<ry^{r-1}(x-y)\quad(0<r<1)$

$rx^{r-1}(x-y)=x^r-y^r=ry^{r-1}(x-y)$

$(r=0, r=1, x=y)$

Lemma 4 ([17]) Let I be a non-empty finite index set, then

$$\left|\bigvee_{i\in I}a_i-\bigvee_{i\in I}b_i\right|\leq\bigvee_{i\in I}|a_i-b_i|,\quad\left|\bigwedge_{i\in I}a_i-\bigwedge_{i\in I}b_i\right|\leq\bigvee_{i\in I}|a_i-b_i|.$$

Lemma 5 Let $x,y\in[0,1]$, $r>1$, then

$|x^r-y^r|\leq r|x-y|$

The proof is easy to carry out from Lemma 3.

Lemma 6 Let the function

$f(x)=(a^m+x^m-1)^{1/m}$, $x\in[c,b]\subset[0,1](c<b)$,

$a\in[0,1]$, $m\geq 1$, $m\neq 0$, then $|f(b)-f(c)|\leq|b-c|$.

Lemma 6 is obtained from a proof of progress of Theorem 5 in paper [8].

Lemma 7 When $m\leq 1$ and $m\neq 0$, let two functions $g(x)=(1-a^m+x^m)^{1/m}$ and $h(x)=(1-x^m+a^m)^{1/m}$, $g(x),h(x)\in[0,1]$, $a\in[0,1]$, $x\in[b,c]\subset[0,1]$, then $|g(b)-g(c)|\leq|b-c|$ and $|h(b)-h(c)|\leq|b-c|$.

The proof of Lemma 7 is similar to Lemma 6.

Lemma 8 (Minkowski's inquality) Let $(a_1,a_2\cdots a_n)$, $(b_1,b_2\cdots b_n)\in R^n$, and $1\leq p<\infty$. Then

$$(\sum_{k=1}^{n}|a_k+b_k|^p)^{1/p}\leq(\sum_{k=1}^{n}|a_k|^p)^{1/p}+(\sum_{k=1}^{n}|b_k|^p)^{1/p}.$$

3.Perturbation of Schweizer-Sklar operators

Proposition 1 Suppose $d_p(A,A')\leq\varepsilon_1$, $d_p(B,B')\leq\varepsilon_2$. A and A' are two fuzzy sets defined on a finite universe contain n elements , B and B' are two fuzzy sets defined on anther finite universe contain k elements, then we have

$$d_p(T_m(A,B),T_m(A',B'))\leq$$

$$\begin{cases}\varepsilon_1+\varepsilon_2,\quad m\in(-\infty,0)\cup(0,\infty);\\ \varepsilon_1+\varepsilon_2,\quad m=0;\\ \sqrt[p]{(\varepsilon_1^p+\varepsilon_2^p)},\quad m=-\infty;\\ \sqrt[p]{(\varepsilon_1^p+\varepsilon_2^p)},\quad m=\infty,\\ 0\ or\ 3\ or\ 4\ element\ in\{a_{1i},a_{2i},b_{1j},b_{2j}\}\ is\ 1.\end{cases}$$

Proof: Suppose

$A=(a_{11},a_{12},\cdots a_{1n})$, $A'=(a_{21},a_{22},\cdots a_{2n})$,

$B=(b_{11},b_{12},\cdots b_{1k})$, $B'=(b_{21},b_{22},\cdots b_{2k})$

$C=T_m(A,B)=\{c_{ij}\}, i=1,2,\cdots n, j=1,2,\cdots k.$,

$C'=T_m(A',B')=\{c'_{ij}\}, i=1,2,\cdots n, j=1,2,\cdots k.$

For $m>1$,

$$|c_{ij}-c'_{ij}|=\left|(a_{1i}^m+b_{1j}^m-1)^{1/m}-(a_{2i}^m+b_{2j}^m-1)^{1/m}\right|$$

$$\leq\frac{1}{m}\left|(a_{1i}^m+b_{1j}^m-1)-(a_{2i}^m+b_{2j}^m-1)\right|(Lemma\ 5)$$

$$\leq\frac{1}{m}(|a_{1i}^m-a_{2i}^m|+|b_{1j}^m-b_{2j}^m|)$$

$$\leq\frac{1}{m}(|m(a_{1i}-a_{2i})|+|m(b_{1j}-b_{2j})|)(Lemma\ 5)$$

$$=|a_{1i}-a_{2i}|+|b_{1j}-b_{2j}|.$$

For $m\leq 1$ and $m\neq 0$,

$$\left|c_{ij}-c'_{ij}\right| = \left|(a_{1i}^m+b_{1j}^m-1)^{1/m}-(a_{2i}^m+b_{2j}^m-1)^{1/m}\right|$$
$$= \left|\begin{matrix}(a_{1i}^m+b_{1j}^m-1)^{1/m}-(a_{1i}^m+b_{2j}^m-1)^{1/m}\\ +(a_{1i}^m+b_{2j}^m-1)^{1/m}-(a_{2i}^m+b_{2j}^m-1)^{1/m}\end{matrix}\right|$$
$$\le \left|(a_{1i}^m+b_{1j}^m-1)^{1/m}-(a_{1i}^m+b_{2j}^m-1)^{1/m}\right|$$
$$+\left|(a_{1i}^m+b_{2j}^m-1)^{1/m}-(a_{2i}^m+b_{2j}^m-1)^{1/m}\right|$$
$$\le \left|b_{1j}-b_{2j}\right|+\left|a_{1i}-a_{2i}\right| (Lemma\ 6).$$

Thus we obtain

$$d_p(T_m(A,A'),T_m(B,B'))$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|c_{ij}-c'_{ij}\right|^p}$$
$$\le\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|+\left|b_{1j}-b_{2j}\right|)^p}$$
$$\le\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|^p}+$$
$$\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|b_{1j}-b_{2j}\right|^p} (Lemma\ 8)$$
$$\le \varepsilon_1+\varepsilon_2$$

For m=0,

$$d(T_m(A,A'),T_m(B,B'))$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|c_{ij}-c'_{ij}\right|^p}$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{1i}\cdot b_{1j}-a_{2i}\cdot b_{2j}\right|^p}$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|a_{1i}\cdot b_{1j}-a_{1i}\cdot b_{2j}+a_{1i}\cdot b_{2j}-a_{2i}\cdot b_{2j}\right|^p}$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(a_{1i}\left|b_{1j}-b_{2j}\right|+b_{2j}\left|a_{1i}-a_{2i}\right|)^p}$$
$$\le\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|b_{1j}-b_{2j}\right|+\left|a_{1i}-a_{2i}\right|)^p}$$
$$\le \varepsilon_1+\varepsilon_2$$

For $m=-\infty$,

$$d_p(T_m(A,B),T_m(A',B'))$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|c_{ij}-c'_{ij}\right|^p}$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|\min\{a_{1i},b_{1j}\}-\min\{a_{2i},b_{2j}\}\right|^p}$$
$$\le\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\max\left\{\left|a_{1i}-a_{2i}\right|,\left|b_{1j}-b_{2j}\right|\right\}^p}$$
$$(Lemma\ 4)$$
$$\le\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|^p+\left|b_{1j}-b_{2j}\right|^p)}$$
$$\le\sqrt[p]{(\varepsilon_1^p+\varepsilon_2^p)}$$

When $m=\infty$, if there is none element in $\{a_{1i},a_{2i},b_{1j},b_{2j}\}$ is 1, the result of the perturbation is 0; if there is one element in $\{a_{1i},a_{2i},b_{1j},b_{2j}\}$ is 1, the result of the perturbation is uncertain, we need to deal with situation like $\left|c_{ij}-c'_{ij}\right|=b_{1j}$; if there two elements in $\{a_{1i},a_{2i},b_{1j},b_{2j}\}$ is 1, the result of the perturbation is uncertain we need to deal with situation like $\left|c_{ij}-c'_{ij}\right|=\left|a_{1i}-b_{2j}\right|$; if there three elements in $\{a_{1i},a_{2i},b_{1j},b_{2j}\}$ is 1, the result of the perturbation is $\sqrt[p]{(\varepsilon_1^p+\varepsilon_2^p)}$; if there four elements in $\{a_{1i},a_{2i},b_{1j},b_{2j}\}$ is 1,the result of the perturbation is 0.

Remark 1 When m=1, the perturbation of Lukasiewicz t-norm is $\varepsilon_1+\varepsilon_2$.

Proposition 2 Suppose $d_p(A,A')\le\varepsilon_1$, $d_p(B,B')\le\varepsilon_2$, A and A' are two fuzzy sets defined on a finite universe contain n elements, B and B' are two fuzzy sets defined on anther finite universe contain k elements, then we have

$$d_p(I_m(A,B),I_m(A',B'))$$
$$\leq\begin{cases}\varepsilon_1+\varepsilon_2, \quad m\in(-\infty,0)\cup(0,\infty);\\ \frac{1}{a}(\varepsilon_1+\varepsilon_2), \quad m=0,\\ a=\min\{a_{1i}\cdot a_{2i}\}, i=1,2,\cdots n\\ \varepsilon_2, \quad m=-\infty,\\ a_{1i}>a_{2i}\ and\ b_{1j}>b_{2j}\ or\ a_{1i}<a_{2i}\ and\ b_{1j}<b_{2j};\\ \varepsilon_2, \quad m=\infty, A=1 and\ A'=1\ or\ A\neq 1 and\ A'\neq 1.\end{cases}$$

Proof: Suppose
$A=(a_{11},a_{12},\cdots a_{1n})$, $A'=(a_{21},a_{22},\cdots a_{2n})$,
$B=(b_{11},b_{12},\cdots b_{1k})$, $B'=(b_{21},b_{22},\cdots b_{2k})$
$C=T_m(A,B)=\{c_{ij}\}, i=1,2,\cdots n, j=1,2,\cdots k.$,
$C'=T_m(A',B')=\{c'_{ij}\}, i=1,2,\cdots n, j=1,2,\cdots k.$
For $m>1$,

$$\left|c_{ij}-c'_{ij}\right|=\left|(1-a_{1i}^m+b_{1j}^m)^{1/m}-(1-a_{2i}^m+b_{2j}^m)^{1/m}\right|$$
$$\leq\frac{1}{m}\left|(1-a_{1i}^m+b_{1j}^m)-(1-a_{2i}^m+b_{2j}^m)\right|(Lemma\ 5)$$
$$\leq\frac{1}{m}(\left|a_{1i}^m-a_{2i}^m\right|+\left|b_{1j}^m-b_{2j}^m\right|)$$
$$\leq\frac{1}{m}(\left|m(a_{1i}-a_{2i})\right|+\left|m(b_{1j}-b_{2j})\right|)(Lemma\ 5)$$
$$=\left|a_{1i}-a_{2i}\right|+\left|b_{1j}-b_{2j}\right|.$$

For $m\leq 1$ and $m\neq 0$,

$$|c_{ij}-c'_{ij}|=|(1-a_{1i}^m+b_{1j}^m)^{1/m}-(1-a_{2i}^m+b_{2j}^m)^{1/m}|$$
$$=|(1-a_{1i}^m+b_{1j}^m)^{1/m}-(1-a_{1i}^m+b_{2j}^m)^{1/m}$$
$$+(1-a_{1i}^m+b_{2j}^m)^{1/m}-(1-a_{2i}^m+b_{2j}^m)^{1/m}|$$
$$=|(1-a_{1i}^m+b_{1j}^m)^{1/m}-(1-a_{1i}^m+b_{2j}^m)^{1/m}|$$
$$+|(1-a_{1i}^m+b_{2j}^m)^{1/m}-(1-a_{2i}^m+b_{2j}^m)^{1/m}|$$
$$\leq|b_{1j}-b_{2j}|+|a_{1i}-a_{2i}|\ (Lemma\ 7).$$

Thus we obtain

$$d_p(I_m(A,A'),I_m(B,B'))$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|c_{ij}-c'_{ij}\right|^p}$$
$$\leq\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|+\left|b_{1j}-b_{2j}\right|)^p}$$
$$\leq\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|^p}$$
$$+\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|b_{1j}-b_{2j}\right|^p}\ (Lemma\ 8)$$
$$\leq\varepsilon_1+\varepsilon_2.$$

For m=0

$$d_p(T_m(A,B),T_m(A',B'))=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|c_{ij}-c'_{ij}\right|^p}$$
$$=\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}\left|\min\left\{1,\frac{b_{1j}}{a_{1i}}\right\}-\min\left\{1,\frac{b_{2j}}{a_{2i}}\right\}\right|^p}$$
$$\leq\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\frac{a_{2i}\left|b_{2j}-b_{1j}\right|+b_{2j}\left|a_{1i}-a_{2i}\right|}{a_{1i}\cdot a_{2i}})^p}$$
$$\leq\frac{1}{a}\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|b_{2j}-b_{1j}\right|+\left|a_{1i}-a_{2i}\right|)^p}$$
$$\leq\frac{1}{a}\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|a_{1i}-a_{2i}\right|^p}+$$
$$\frac{1}{a}\sqrt[p]{\frac{1}{nk}\sum_{i=1}^{n}\sum_{j=1}^{k}(\left|b_{1j}-b_{2j}\right|^p}\ (Lemma\ 8)$$
$$\leq\frac{1}{a}(\varepsilon_1+\varepsilon_2)$$
$$a=\min\{a_{1i}\cdot a_{2i}\}, i=1,2,\cdots n$$

For $m=-\infty$, if $a_{1i}\leq b_{1j}, a_{2i}\leq b_{2j}$, then $d_p(c_{ij},c'_{ij})=0$; if $a_{1i}>b_{1j}, a_{2i}>b_{2j}$, then $d_p(I_m(A,A'),I_m(B,B'))\leq\varepsilon_2$; if $a_{1i}\leq b_{1j}, a_{2i}>b_{2j}$ or $a_{1i}>b_{1j}$, $a_{2i}\leq b_{2j}$, the result of $d_p(I_m(A,A'),I_m(B,B'))$ is uncertain.

For $m=\infty$, if $a_{1i}\neq 1, a_{2i}\neq 1$, then $d_p(I_m(A,A'),I_m(B,B'))=0$; if $a_{1i}=1, a_{2i}=1$, then $d_p(I_m(A,A'),I_m(B,B'))\leq\varepsilon_2$; if $a_{1i}\neq 1, a_{2i}=1$ or $a_{1i}=1, a_{2i}\neq 1$, the result of $d_p(I_m(A,A'),I_m(B,B'))$ is uncertain.

Remark 2 When m=1, the perturbation of Lukasiewicz implication is $\varepsilon_1+\varepsilon_2$.

4.Robustness of full implication triple I algorithms

In this section, we will study the robustness of triple I algorithms based on Schweizer-Sklar parameterized operators.

Theorem 1 Suppose $d_p(A,A')\le\varepsilon_1$, $d_p(B,B')\le\varepsilon_2$, $d_p(A^*,A'^*)\le\varepsilon_3$, B^* and B'^* are the $\to_m$-type triple I solutions for FMP, then

$$d_p(B^*,B'^*)\le \begin{cases}\varepsilon_1+\varepsilon_2+\varepsilon_3, \quad m\in(-\infty,0)\cup(0,\infty),\\ (A\to_m B)^m+(A^*)^m-1>0,\\ (A'\to_m B')^m+(A'^*)^m-1>0;\\ \frac{1}{a}(\varepsilon_1+\varepsilon_2)+\varepsilon_3, m=0(a=\min\{a_{1i}\cdot a_{2i}\}, i=1,2,\cdots n)\\ A^*>0\ and\ A>B, A'^*>0\ and\ A'>B'\\ \sqrt[p]{(\varepsilon_2^p+\varepsilon_3^p)}, \quad m=-\infty,\\ A^*>0\ and\ A>B, A'^*>0\ and\ A'>B'\end{cases}$$

$$\begin{cases}\sqrt[p]{(\varepsilon_2^p+\varepsilon_3^p)}, \quad m=\infty,\\ A^*\vee(A\to_m B)=1 and\ A'^*\vee(A'\to_m B')=1\\ 0, \quad m=\infty,\\ A^*\vee(A\to_m B)\ne 1 and\ A'^*\vee(A'\to_m B')\ne 1\end{cases}$$

Proof: For $m\in(-\infty,0)\cup(0,\infty)$,

let $C=A\to_m B$, $C'=A'\to_m B'$,

$(C^m+(A^*)^m-1)^{1/m}=\{d_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$,

$(C'^m+(A'^*)^m-1)^{1/m}=\{d'_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$.

Then we have $d_p(C,C')\le\varepsilon_1+\varepsilon_2$ by Proposition 2.

Moreover, we have

$$d_p(B^*,B'^*)\le\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}$$
$$\le\varepsilon_1+\varepsilon_2+\varepsilon_3$$

by Lemma1 、Proposition 1 and Lemma 4.

When m=0,

let $C=A\to_m B$, $C'=A'\to_m B'$,

$(C\square A^*)=\{d_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$,

$(C'\square A'^*)=\{d'_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$.

Then we have $d_p(C,C')\le\frac{1}{a}(\varepsilon_1+\varepsilon_2)$, $a=\min\{a_{1i}\cdot a_{2i}\}, i=1,2,\cdots n$ by Proposition 2, and

$$d_p(B^*,B'^*)=\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}$$
$$\le\frac{1}{a}(\varepsilon_1+\varepsilon_2)+\varepsilon_3$$

by Lemma1 and Proposition 1.

When $m=-\infty$,

let $C=A\to_m B$, $C'=A'\to_m B'$,

$(C\wedge A^*)=\{d_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$,

$(C'\wedge A'^*)=\{d'_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$.

Then we have $d_p(C,C')\le\varepsilon_2$ by Proposition 2, and

$$d_p(B^*,B'^*)=\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}$$
$$\le\sqrt[p]{\varepsilon_2^p+\varepsilon_3^p}$$

by Lemma1 and Proposition 1.

When $m=\infty$,

let $C=A\to_m B$ $C'=A'\to_m B'$,

$(C\wedge A^*)=\{d_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$,

$(C'\wedge A'^*)=\{d'_{qi}\}, q=1,2,\cdots,nk, i=1,2,\cdots,n$

Then we have $d_p(C,C')\le\varepsilon_2$ by Proposition 2, and

$$d_p(B^*,B'^*)=\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}$$
$$\le\sqrt[p]{\varepsilon_2^p+\varepsilon_3^p}$$

by Lemma1 and Proposition 1.

Theorem 2 Suppose $d_p(A,A')\le\varepsilon_1$, $d_p(B,B')\le\varepsilon_2$, $d_p(B^*,B'^*)\le\varepsilon_3$ and A^* and A'^* are the $\to_m$-type triple I solution for FMT, then

$$d_p(A^*,A'^*)\le\begin{cases}\varepsilon_1+\varepsilon_2+\varepsilon_3, m\in(-\infty,0)\cup(0,\infty);\\ \frac{1}{b}(\varepsilon_1+\varepsilon_2+\varepsilon_3), m=0,(b=\min\{b_{1j},b_{2j}\}, j=1,2,\cdots k;)\\ \varepsilon_3, m=-\infty.\end{cases}$$

For $B^*<A\to_m B$ and $B'^*<A'\to_m B'$,

when $m<\infty$,

$$d_p(A^*,A'^*)=\begin{cases}\varepsilon_3, & A\rightarrow_m B=1 \; and \; A'\rightarrow_m B'=1\\ 0, & A\rightarrow_m B\neq 1 \; and \; A'\rightarrow_m B'\neq 1\end{cases}$$

Proof: For $m\in(-\infty,0)\cup(0,\infty)$,
let $C=A\rightarrow_m B$, $C'=A'\rightarrow_m B'$,
$(1-C^m+(B^*)^m)^{1/m}=\{d_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$,
$(1-C'^m+(B'^*)^m)^{1/m}=\{d'_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$.
Then we have $d(C,C')\le\varepsilon_1+\varepsilon_2$ by Proposition 2, and

$$d(A^*,A'^*)\le\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}\le\varepsilon_1+\varepsilon_2+\varepsilon_3$$

by Lemma 2 , Proposition 2 and Lemma 4.
When m=0,

let $C=A\rightarrow_m B$, $C'=A'\rightarrow_m B'$,

$$(\frac{B^*}{C})=(\frac{B^*}{A\rightarrow B})=\frac{A\cdot B^*}{B}=\{d_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$$
,
$$(\frac{B'^*}{C'})=(\frac{B'^*}{A'\rightarrow B'})=\frac{A'\cdot B'^*}{B'}=\{d'_{qi}\}, q=1,\cdots,nk, i=1,\cdots,n$$

Then we have $d(A\cdot B^*,A'\cdot B'^*)\le\varepsilon_1+\varepsilon_3$ by Proposition 1, and

$$d(A^*,A'^*)=\sqrt[p]{\frac{1}{n^2k}\sum_{q=1}^{nk}\sum_{i=1}^{n}\left|d_{qi}-d'_{qi}\right|^p}$$

$$\le\frac{1}{b}(\varepsilon_1+\varepsilon_2+\varepsilon_3),$$

$$b=\min\{b_{1j},b_{2j}\}, j=1,2,\cdots k;$$

by Lemma 2 and Proposition 2.

When $m=-\infty$ and $m=\infty$, we have $d_p(A^*,A'^*)\le d_p(B^*,B'^*)=\varepsilon_3$ by Lemma 2 and Lemma 4.

5.Conclusion

In this paper, the perturbation of Schweizer-Sklar parameterized family of operators was investigated. The robustness of triple I algorithms based on Schweizer-Sklar operators were studied. We proved that triple I algorithms based on Schweizer-Sklar t-norms have robustness for $m\in(-\infty,\infty)$, and robustness with special conditions for $m=-\infty$ and $m=\infty$. These conclusions will provide a reliable theoretical basis for fuzzy reasoning.

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Conflicts of interest

The authors have no conflicts of interest to declare.

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